On The Functional Equations In Rectilinear Embedding

Enumerating Some Types Of Quadrangulations

Pan liyan [1], Liu yanpei [2]
Department of mathematics
Beijing Jiaotong University
Beijing, China
[1]panliyan@bjtu.edu.cn, [2]ypliu@bjtu.edu.cn

***Abstract*—This paper provides quartic functional equations satisfied by the enumerating functions of rooted planar near-quadrangulations and cubic enunfunctions of rooted nonseparable outerplanar quadrangulations, explicit formulae for such these quadrangulations with map fundamental parameters are derived respectively after employing Lagrangian inversion. Quadrangulations and 4-regular maps (or quartic maps as some scholars called them) are very important, the usage can be seen for rectilinear embedding in VLSI, for the Gaussian crossing problem in graph theory, for the knot problem in topology, and for the enumeration of some other kinds of maps.**

Keywords-Quadrangulation;Quadrangulation;Lagrangian inversion; Enumerating function; VLSI

I. INTRODUCTION

A map considered here is a 2-cell embedding for a graph on the surface. If the surface is the plane, or the sphere, the map is called a planar map. A rooted map is a map M such that one edge on the boundary of the outer face is selected and oriented so that the outer face is on the right hand side of it when one moves along the boundary with the direction. The selected edge is root-edge; its tail vertex is root-vertex; the outer face is root-face. Edges (vertices) are external or internal according as they do or do not lie on the boundary of the root-face. Without loss of generality, the root-face is chosen as the infinite face.

Quadrangulations and 4-regular maps have been investigated by many scholars such as Tutte [1], Mullin and Schellenberg [2], and Liu [3, 4]. In [5], William G.Brown obtained the number of quadrangular dissections of the disc in terms of external and internal vertices.

A planar quadrangulation is a map on the plane such that each face is 4-gon. A map is a near quadrangulation if its each inner face is quadrangle except possibly for the root-face. By an outerplanar map, we mean a map in which all the vertices are on the boundary of the infinite face. An outerplanar quadrangulation is said to be nonseparable if its underlying graph has no cut-vertex. An outerplanar near quadrangulation is, in fact, a quadrangulation on the disc, if an outerplanar quadrangulation has its root-face boundary on a circuit. A map is said to be Hamiltonian if all the vertices of the map are on one circuit. A Halin map is a planar one oriented by a tree. It is formed by joining all the leaves of a tree consecutively along the facial walk. We call a Halin map (k, 3)-regular if all of its vertices are k-valent except possibly those on the root-face. The importance of Halin maps has been discovered in graph embedding theory as well as enumerating of maps.

A map denoted by $M=(\chi_{\alpha,\beta},\rho)$, where $\chi_{\alpha,\beta}(X)=\sum_{x\in X} Kx$, X is a finite set and ρ is a basic permutation on X. The root, the root-vertex, root-edge and root-face are denoted by $r(M)$, $v_r(M)$, $e_r(M)$ and $f_r(M)$ respectively. If a map has a single edge which is not a loop, then it is called a link map, denoted by $L=(Kr,(r)(\alpha\beta r))$. For convenience, the notations and terminologies not mentioned here can be seen in [3].

II. FUNCTIONAL EQUATIONS FOR Q_1, Q_{2e}, Q_{nq}

Let Q_1 be the set of all rooted planar near-quadrangulations without 2-boundary ones; Q_{2e} be the subset of Q_1 and for any $M\in Q_{2e}$, M is 2 edge-connected; Q_{nq} be the set of all general rooted planar near-quadrangulations including 2-boundary ones. For a map M, the root, the root-vertex, the root-edge and the root-face are denoted by $r(M)$, $v_r(M)$ and $e_r(M)$, $f_r(M)$ respectively. Since the valency of all faces of planar near quadrangulations is four probably except the root-face, the root-face valency of any planar near quadrangulation has to be even. We define generating functions for Q_i, $i=1,2e$, nq, rvn(rooted vertex is nonseparable).

$$f_i(x,y)=\sum_{M\in Q} x^{2m(M)}y^{n(M)}z^{t(M)}$$

Where $2m(M)$, $n(M)$ and $t(M)$ are the valency of the root-face, the size and the number of non-rooted vertices. Further, we write that

$$G_i(x)=f_i(x,1),\ F_i(y)=f_i(1,y), h_i(x,z)=f_i(x,1,z)$$

For $i=1,2e$, nq, rvn as some special enunfunctions.

In this section, we obtain main results as follows:

- Theorem II.A. The enumerating function $f_1(x,y)$ satisfies the following equation

$$x^2yf_1^4-(1-x^{-2}y)f_1^3+(1-3x^{-2}y)f_1^2+3x^{-2}yf_1-x^2y=0 \qquad (1)$$

Furthermore,

$$F_1(y)=1+\sum_{n\ge1}A_n y^n,\ A_n=\sum_{k\ge1}^{\left\lfloor\frac{n+2}{3}\right\rfloor}\frac{(n-1)!(2n-4k+4)!}{(k-1)!\left(n-k+1\right)!(n-k+2)!\left(n-3k+2\right)!};$$

$$G_1(x)=1+\sum_{m\ge1}A_m x^{2m},\ A_m=\sum_{0\le i\le m-1}\frac{(2m)!(3m-2i-3)!}{i!\left(2m-i\right)!(m-i-1)!\left(2m-i-1\right)!}$$

- Theorem II.B. The enumerating function $f_{2e}(x,y)$ satisfies the following equation

$$(x^2-y)f_{2e}^{\ 3}-(x^2+x^4y-3y)f_{2e}^{\ 2}-3yf_{2e}+y=0 \quad (2)$$

Moreover, for $i+j\le m$, $0\le i\le m+2$ $0\le j\le m-2$

$$F_{2e}(y)=1+\sum_{n\ge1}B_n y^n,\ G_{2e}(x)=1+\sum_{m\ge1}B_m x^{2m};$$

$$B_n=\sum_{k\ge1}^{\left\lfloor\frac{n-1}{3}\right\rfloor}\frac{(n-1)!(n-k-2)!}{k!\left(n-k\right)!(2k-1)!(n-3k-1)!};$$

$$B_m=\sum_{i,j}\frac{3^{m-i-j}2^i(m+2)!(m-2)!(2m-i-j)!}{i!\,j!(2m-2i+4)!!(2m-2j-4)!!(m-i-j)!}$$

- Theorem II.C. The enumerating function $h_{\mathrm{nq}}(x,z)$ satisfies the following equation

$$x^4zh_{\mathrm{nq}}^{\ 2}+(1-x^2)h_{\mathrm{nq}}+x^2-x^2H_{\mathrm{nq}}-1=0$$

Where H_{nq} is the coefficient of x^2 in $h_{\mathrm{nq}}(x,z)$. And the explicit solution of (2) is

$$h_{\mathrm{nq}}(x,z)=\sum_{n\ge m\ge1}C_{m,n}x^{2m}z^n \quad C_{m,n}=\frac{3^{n-m}(2m)!}{m!\left(m-1\right)!}\frac{(2n-m-1)!}{\left(n-m\right)!(n+1)!}$$

III. ESTABLISHMENT OF THE EQUATIONS FOR f_1, f_{2e}, h_{nq}

For two maps M_1 and M_2 with their respective roots $r_1=r(M_1)$ and $r_2=r(M_2)$. The map $M=M_1\cup M_2$, provided $M_1\cap M_2=\{v\}$ with $v=v_{r_1}=v_{r_2}$ is defined to have its root, root-vertex and root-edge are as the same as those of M_1, but the root-face is the composition of $f_{r_1}(M_1)$ and $f_{r_2}(M_2)$, where $f_{r_i}(M_i)$ is the root-face of M_i ($i=1,2$). The operation for getting M from M_1 and M_2 is called *1v-addition* and is denoted b $M=M_1\dot{+}M_2$.

Further, for two sets of maps $M^{(1)}$ and $M^{(2)}$, the set of maps $M^{(1)}\odot M^{(2)}=\{M_1\dot{+}M_2 \mid M_i\in M^{(i)}, i=1,2\}$ is said to be the *1v-production of* $M^{(1)}$ and $M^{(2)}$.

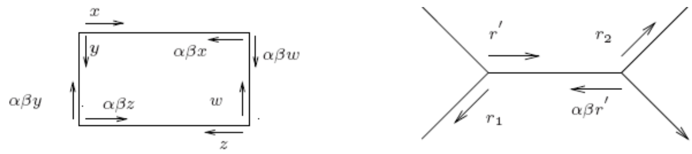

Fig. 1 Quadrangle and $Q_1^{(\mathrm{I})}$

Here we consider $Q_1(x,y)$ first, which can be partitioned into three parts:

1) $Q_1^{(0)}=\{v\}$, v is the vertex map;

2) $Q_1^{(\mathrm{I})}=\{M \mid e_r(M)$ is an isthmus$\}$.
The link map $L=(Kr,(r)(\alpha\beta r))$ is included;

3) $Q_1^{(\mathrm{II})}=\{M \mid e_r(M)$ belongs to a simple circuit$\}$.

$Q_{\mathrm{nq}}(x,y)$ follows the similar pattern of this partition.

- Lemma III.A. Let $Q^{(\mathrm{I})}_{\langle1\rangle}=\{M-a \mid \forall M\in Q_1^{(\mathrm{I})}\}$ where $a=e_r(M)$ is the root-edge of M, then $Q^{(\mathrm{I})}_{<1>}=Q_1\times Q_1$ (where $\times$ presents Descartes product between sets).

Proof**:** Because $\forall M\in Q_1^{(\mathrm{I})}$, the root-edge a of M is a cut edge, $M-a=M_1\dot{+}M_2, M_1, M_2\in Q_1$. That implies

$M-a\in Q_1\times Q_1$. Hence $Q^{(\mathrm{I})}_{<1>}\subseteq Q_1\times Q_1$

Conversely, for $\forall M\in Q_1\times Q_1$, we have $M=M_1\dot{+}M_2$. $M_1, M_2\in Q_1$. The only map $M'=(X',P')$ can be obtained by adding an edge Kr', which connects $v_r(M_1)$ and $v_r(M_2)$ (See Figure 1). Then M' is also outerplanar quadrangulation, and Kr' is a cut edge, so $M'\in Q_1^{(\mathrm{I})}$, $M=M'-a\in Q^{(\mathrm{I})}_{<1>}$ .This means that $Q_1\times Q_1\subseteq Q^{(\mathrm{I})}_{\langle1\rangle}$.

From Lemma III.A, we obtained the contribution of $Q_1^{(\mathrm{I})}$ to Q_1 is $f_1^{(\mathrm{I})}=x^2yf_1^{\ 2}$ (3)

Where x^2 presents the contribution of the root-edge to the root-face boundary of a map M in $Q_1^{(\mathrm{I})}$.

- Lemma III.B. $Q_1 = \sum_{k\geq 0} Q_{\mathrm{rvn}}^{\odot k}$ Where $Q_{\mathrm{rvn}} = \{M \mid \forall M \in Q_{\mathrm{rvn}} \subset Q_1, v_r(M)$ is nonseparable$\}$.

Proof $\forall M \in Q_1$, it is easily to see that M has k components, M_i ($1 \leq i \leq k$) and $M_i \in Q_{\mathrm{rvn}}$, such that $M = M_1 \dot{+} M_2 + \dot{+} M_3 \ldots + \dot{+} M_k$, hence $M \in Q_{\mathrm{rvn}}^{\odot k}$.

From Lemma III.B, $f_1 = \sum_{k\geq 0} f_{\mathrm{rvn}}^k = \dfrac{1}{1-f_{\mathrm{rvn}}}$, by simplification we obtained that $f_{\mathrm{rvn}} = \dfrac{f_1 - 1}{f_1}$.

For a map $M \in Q_1^{(\mathrm{II})}$, $a = kr$ is the root-edge of M, let $M \hat{-} a$ be the map obtained by deleting the edge a, such that the root of $M \hat{-} a$ is $(P\alpha\beta Pr)$, where $(Pr, P\alpha\beta Pr, (P\alpha\beta)^2 Pr, a)$ is the quadrangle incident to a.

- Lemma III.C. Let $Q_{\langle 1\rangle}^{(\mathrm{II})} = \{M - a \mid \forall M \in Q_1^{(\mathrm{II})}\}$.

Then $Q_{\langle 1\rangle}^{(\mathrm{II})} = (Q_1 - Q_1^{(0)}) \times Q_{\mathrm{rvn}}^2$

Proof: $\forall M^* = (X^*, P^*) \in (Q_1 - Q_1^{(0)}) \times Q_{\mathrm{rvn}}^2$, for the root-vertex v_{r^*} is a cut vertex, we may obtained a map $M = M^* \hat{+} a^*$ by join an edge $a^* = (v_{(P\alpha\beta)^{2m-1}r^*}, v_{(P\alpha\beta)^2 r^*})$ where the root-face of M^* is $(r^*, P\alpha\beta r^*, \cdots\cdots, (P\alpha\beta)^{2m-1}r^*)$. Since a^* is an edge on a circuit, so $M = M^* \hat{+} a^* \in Q_1^{(\mathrm{II})}$ (See Figure 2), it is easily checked that $M^* = M \hat{-} a^*$.

By considering that the valency of the root-face of a map in $Q_1^{(\mathrm{II})}$ is 2 less than that of the corresponding map in $(Q_1 - Q_1^{(0)}) \times Q_{\mathrm{rvn}}^2$, we have $Q_1^{(\mathrm{II})}$ to Q_1 is $f_1^{(\mathrm{II})} = x^{-2} y (f_1 - 1) f_{\mathrm{rvn}}^2$. For $f_{\mathrm{rvn}} = \dfrac{f_1 - 1}{f_1}$, we get that

$$f_1^{(\mathrm{II})} = x^{-2} y (f_1 - 1)\left(\frac{f_1 - 1}{f_1}\right)^2 \tag{4}$$

Since $f_1 = f_1^{(0)} + f_1^{(\mathrm{I})} + f_1^{(\mathrm{II})}$, from $f_1^{(0)} = x^0 y^0 = 1$ and (3),(4), we have

$$f_1 = 1 + x^2 y f_1^2 + x^{-2} y (f_1 - 1)\left(\frac{f_1 - 1}{f_1}\right)^2$$

Multiply by f_1^2 the two sides, Theorem II.A (1) follows from some rearrangement.

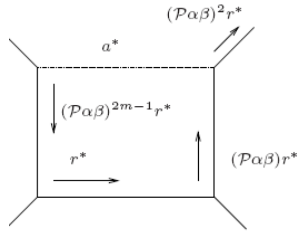

Fig. 2 $Q_1^{(\mathrm{II})}$

- Lemma III.D. Let $Q_{(\mathrm{nq})}^{(\mathrm{I})} = \{M \bullet e_r(M) \mid M \in Q_{\mathrm{nq}}^{(\mathrm{I})}\}$.

Then $Q_{(\mathrm{nq})}^{(\mathrm{I})} = Q_{\mathrm{nq}} \odot Q_{\mathrm{nq}}$

Proof $\forall M \in Q_{\mathrm{nq}} \odot Q_{\mathrm{nq}}$, there exists $M = M_1 \dot{+} M_2$, M_1, $M_2 \in Q_{\mathrm{nq}}$. After splitting the root-vertex of M, it will result in a map M' in $Q_{\mathrm{nq}}^{(\mathrm{I})}$ such that $M = M' \bullet e_r(M')$, this procedure is reversible.

From Lemma III.D, we see that the contribution of $Q_{\mathrm{nq}}^{(\mathrm{I})}$ to Q_{nq} is $h_{\mathrm{nq}}^{(\mathrm{I})} = x^2 z h_{\mathrm{nq}}^2$.

- Lemma III.E. Let $Q_{\langle\mathrm{nq}\rangle}^{(\mathrm{II})} = \{M - e_r(M) \mid M \in Q_{\mathrm{nq}}^{(\mathrm{II})}\}$,

then $Q_{\langle\mathrm{nq}\rangle}^{(\mathrm{II})} = Q_{\mathrm{nq}} - v - Q_{\mathrm{nq}2}$, where

$Q_{\mathrm{nq}2} = \{M \mid \forall M \in Q_{\mathrm{nq}2} \subset Q_{\mathrm{nq}}$ the valency of M is $2\}$.

Proof: $\forall M \in Q_{\langle\mathrm{nq}\rangle}^{(\mathrm{II})}$, $M = M^* - e_{r^*}(M^*)$, $M^* \in Q_{\mathrm{nq}}^{\mathrm{II}}$. Since $e_{r^*}(M^*)$ is on a circuit, so the valency of $f_r(M)$ is not less than 4. Conversely, the only map M can be constructed by adding an edge $(v_{(P\alpha\beta)^{2m-1}r^*}, v_{(P\alpha\beta)^2 r^*})$ in M^*.

Let H_{nq} be the coefficient of x^2 in $h_{\mathrm{nq}}(x,z)$.

From Lemma III.E, we have

$$h_{\mathrm{nq}}^{(\mathrm{II})} = x^{-2}(h_{\mathrm{nq}} - 1 - x^2 H_{\mathrm{nq}}) \tag{5}$$

Since $h = h_{\mathrm{nq}} = h_{\mathrm{nq}}^{(0)} + h_{\mathrm{nq}}^{(\mathrm{I})} + h_{\mathrm{nq}}^{(\mathrm{II})}$, from $h_{\mathrm{nq}}^{(0)} = x^0 z^0 = 1$,

$h_{\mathrm{nq}}^{(\mathrm{I})} = x^2 z h_{\mathrm{nq}}^2$ and (5), we get

$$h_{nq} = 1 + x^2 z h_{nq}^2 + x^{-2}(h_{nq} - 1 - x^2 H_{nq})$$

By rearranging the terms, we soon find the Theorem II.C (2).

IV. PARAMETRIC EXPRESSIONS FOR $f_1(1, y)$

Although from (1) it is allowed to find f_1 directly by using Lagrangian inversion, because of the equation is quadruple, the result is rather complicated for usage. In order to find the enumerating explicit expressions, we have to do some transformation and find their parametric expressions firstly.

For the generating function $f = f_1$ satisfies (1), then $F = f_1(1, y)$ satisfies

$$F^4 + (1 - \frac{1}{y})F^3 + (\frac{1}{y} - 3)F^2 + 3F = 1$$

Because the left side of the equation has the following decomposition, so we write

$$F^3 + (2 - \frac{1}{y})F^2 - F + 2 = \frac{1}{1-F} = 1 + F + F^2 + F^3 + \frac{F^4}{1-F}$$

Which is in fact the form as

$$(\frac{1-F}{F})^2 = \frac{1}{y} + \frac{F^2}{1-F} \tag{6}$$

- Lemma IV.A. The generating function $F = f_1(1, y)$ has the parametric expressions as follows:

$$\begin{cases} F = \dfrac{1}{1-t} \\ y = \dfrac{t(1-t)}{t^3(1-t)+1} \end{cases} \tag{7}$$

Proof: From equation (6), we introduce one parameter for F, let $t = \frac{1-F}{F}$ then the parametric expression is derived.

By [6], according to the duality, the (2) have the following parametric expressions:

$$\begin{cases} z = \theta\dfrac{(2-3\theta)}{4} \\ h = \dfrac{x^2 - 1 + (1+ux^2)(1-vx^2z)^{\frac{1}{2}}}{2x^4 z} \end{cases}$$

Where, $\begin{cases} -2 = -vz + 2u, 4z + 1 = -2uvz + u^2 \\ -u^2 vz = -4y(1+q), \theta = u + 1 \end{cases}$

V. EXPLICIT FORMULA FOR $f_1(1, y)$

In this section, we enumerate the explicit expressions of the enumerating functions $F = f_1(1, y)$ by employing Lagrangian inversion based on the results as described above.

For the sake of brevity, let

$$\partial_x^i = \begin{cases} \left.\dfrac{1}{i!}\dfrac{d^i}{dx^i}\right|_{x=0} & i \ge 0 \\ \left.x^{-i}\right|_{x=\infty} & i < 0 \end{cases}$$

Which is called the coefficient operator, or the partial, here we use the notation

$$\partial_x^i f = [x^i] f$$

In (7), we can see that $t = y\varphi(t)$ where $\varphi(t) = t^3 + \frac{1}{1-t}$. We get

$$[y^n]F = \frac{1}{n}[t^{n-1}](\varphi^n(t) \bullet \frac{dF}{dt})$$

by applying Lagrangian inversion with one parameter,

$$\begin{aligned} [y^n]F &= \frac{1}{n}[t^{n-1}](\frac{1}{(1-t)^{n+2}}(1 + t^3(1-t))^n) \\ &= \frac{1}{n}[t^{n-1}]\sum_{k\ge0}^{n}\binom{n}{k} t^{3k}(1-t)^{k-n-2} \\ &= \frac{1}{n}[t^{n-3k-1}]\sum_{k\ge0}^{n}\binom{n}{k}(1-t)^{k-n-2} \\ &= \frac{1}{n}\sum_{k\ge0}^{\left\lfloor\frac{n-1}{3}\right\rfloor}\binom{n}{k}\binom{2n-4k}{n-3k-1} \end{aligned}$$

So $$F(y) = \frac{1}{n}\sum_{k\ge0}^{\left\lfloor\frac{n-1}{3}\right\rfloor}\binom{n}{k}\binom{2n-4k}{n-3k-1}y^n$$

which is equivalent to Theorem II.A. According to this theorem, we compute the coefficients A_n of $F(y)$, which are shown in TABLE I with respect to edges within 20 edges.

By Euler Formula, for any planar near quadrangulation, if the valency of root face is $2m$ and the number of non-root vertices is n, we can derived the edge number is $2n - m$. Therefore, h_{nq} can be expanded as

$$f_{nq} = \sum_{n\ge m\ge1} C_{m,n} x^{2m} z^n y^{2n-m} \tag{8}$$

- Corollary V.A. (Reference [1])

Let T be the enumerating function of rooted planar trees, then

$$T=\sum_{n\ge1}\frac{(2n)!}{n!(n+1)!}y^n$$

TABLE I. NUMBERS OF OUTERPLANAR QUADRANGULATION

x	A_n
0	1
1	1
2	2
3	5
4	15
5	48
6	160
7	552
8	1953
9	7044
10	25806
11	95765
12	359216
13	1359767
14	5187754
15	19927572
16	77006278
17	299153776
18	1167638982
19	4576743657
20	18007689546

Proof: Notice that if $m=n$ in $C_{m,n}$, then the expression of T is obtained.

Let $F_{\mathrm{nq}}(y)=\sum_{n\ge0}C_n y^n$ denote the rooted general planar near-quadrangulation with edge as a parameter, from (8), we see that

$$C_n=\sum_{n\ge m\ge\left\lceil\frac{n+1}{2}\right\rceil}\frac{3^{n-m}(4m-2n)!}{(2m-n-1)!(2m-n)!}\frac{(n-1)!}{(n-m)!(m+1)!}y^n$$

Further, let $N_{\mathrm{q}}(n)$ be the number of rooted general planar quadrangulation with n edges, $N_{\mathrm{nq2}}(n+1)$ be the number of Q_{nq2}, rooted planar near quadrangulation with $n+1$ edges and the valency of the root-face is 2.

Based on the discussion above, so the corollary follows.

- Corollary V.B. $F_{\mathrm{nq2}}=\sum_{n\ge1}(C_n-A_n)y^n$
- Corollary V.C. $N_{\mathrm{nq2}}(n+1)=N_{\mathrm{q}}(n)$

Proof: For $\forall M\in Q_{\mathrm{nq2}}$ can be obtained by replacing the root-edge with an additional edge on maps in Q_{q}, forming a multi-edge to the root-edge as the root-face boundary. On the other hand, $\forall M\in Q_{\mathrm{q}}$, M can be got by shrinking the multi-edge on the boundary of maps in Q_{nq2}.

VI. FUNCTIONAL EQUATIONS FOR Q_0, Q_d

Let Q_0 be the set of all rooted nonseparable outerplanar quadrangulations and Q_d be the set of all rooted near quadrangulations on the disc. We define two generating functions for Q_0 and Q_d:

$$f_0(x,y,z)=\sum_{M\in Q_0}x^{m(M)}y^{n(M)}\qquad g_d(x,y)=\sum_{M\in Q_d}x^{m(M)}y^{n(M)}$$

Where $m(M)$, $n(M)$ is the number of edges, the valency of the root-face of M.

First we make a partition on Q_0 and Q_d, i.e., Q_I, Q_{II}, Q_{III}, Q_{IV}. We denote the two vertices follow the root-vertex v_r are $v_1=\rho\alpha\beta v_r$, $v_2=(\rho\alpha\beta)^2v_r$, the two edges follow the root-edge e_r are $e_1=\rho\alpha\beta e_r$, $e_2=(\rho\alpha\beta)^2e_r$ respectively when we move along the root-edge direction on the root-face, then the four cases are :

Case 1. Q_I is a quadrangle;

Case 2. $\forall M\in Q_{II}\subseteq Q-Q_I$, the valency of v_1 and v_2 are 2;

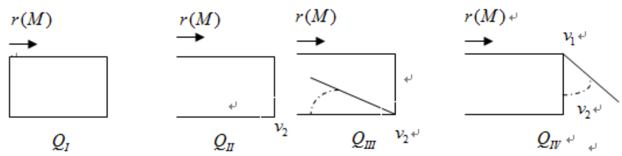

Fig. 3 Combinatorially distinct rooted cases of Q_0

Case 3. $\forall M\in Q_{III}\subseteq Q-Q_I-Q_{II}$, the valency of v_1 is 2;

Case 4. $\forall M\in Q_{IV}$, the valency of v_1 is greater than 2.

A. *The enumeration of* Q_0

For Q_d can be divided into the same parts as Q_0 except in case Q_{IV}, so we consider Q_0 first. Here the enunfunctions for Q_i are denoted by f_i (i=I,II,III,IV). From Fig. 3, there are three choices of taking the root in Q_{II}, two choices in Q_{III}, only one choice in Q_I and Q_{IV}, it is easily to see that

$$Q_0=Q_I+3Q_{II}+2Q_{III}+Q_{IV}$$

For two maps M_1 and M_2 with their respective roots $r_1=r(M_1)$ and $r_2=r(M_2)$ .The map $M=M_1\cup M_2$, provided $M_1\cap M_2=\{v\}$ with $v=v_1=v_2$ is defined to have its root, root-vertex and root-edge is the same as those of M_1 ,but the root-face is the composition of $f_{r_1}(M_1)$ and $f_{r_2}(M_2)$, where $f_{r_i}(M_i)$ is the root-face of $M_i(i=1,2)$ .The operation for getting M from M_1 and M_2 is called $1v$ -addition and is denoted by $M=M_1\dot{+}M_2$ .Further, for two sets of maps $M^{(1)}$ and $M^{(2)}$, the set of maps $M^{(1)}\otimes M^{(2)}=\{M_1\dot{+}M_2 \mid M_i\in M^{(i)},i=1,2\}$ is said to be $1v$ -production of $M^{(1)}$ and $M^{(2)}$.

- Lemma VI.A.1 See Fig. 4
  Let $Q_{<II>}=\{M-e_r-e_1-e_2 \mid \forall M\in Q_{II}\}$, then $Q_{<II>}=Q_0$.

  From LemmaA.1, we obtained the enunfunction of Q_{II} is

$$f_{II}=3f_{Q_{II}}=3x^3y^2f_0 \tag{9}$$

  where y^2 presents the contribution of three adding edges to the root-face boundary.
- Lemma VI.A.2 See Fig. 5
  Let $Q_{<III>}=\{M-e_r-e_1 \mid \forall M\in Q_{III}\}$, then $Q_{<III>}=Q_0\otimes Q_0$

  On the basis of lemma A.2, the enunfunction of Q_{III} is

$$f_{III}=2f_{Q_{III}}=2x^2f_0^2 \tag{10}$$

In case 4, as for the set Q_{IV} , it may be further partitioned into another two parts as $Q_{IV}=Q_{IV1}+Q_{IV2}$.

Subcase1: For any $M^*\in Q_{IV1}$, let $M=M^*-e_r^*$, then M can be separated into three submaps: Q_1 , Q_2 and L (link map), two cut-vertices are v_{r_1} and v_{r_2} (see Fig. 6), v_{r_i} is the root-vertex of Q_i , $Q_i\in Q_0$ ($i=1,2$).

Subcase2: For any $M^*\in Q_{IV2}$, we substitute Q_3 for the link map L , three cut-vertices are v_{r_1} and v_{r_3} (see Fig. 7), v_{r_i} is the root-vertex of Q_i , $Q_i\in Q_d$ (or Q_0),($i=1,2,3$).

- Lemma VI.A.3
  Let $Q_{<IV2>}=\{M-e_r \mid \forall M\in Q_{IV2}\}$, then $Q_{<IV2>}=Q_0\otimes Q_0\otimes Q_0$
- Lemma VI.A.4
  Let $Q_{<IV1>}=\{M-e_r \mid \forall M\in Q_{IV1}\}$, then $Q_{<iv1>}=Q_0\otimes Q_0\otimes L$

These two lemmas enable us to get the enunfunction for case4 as

$$f_{IV}=f_{IV1}+f_{IV2}=xy^{-2}(f_L\bullet f_0^2+f_0^3)=x^2f_0^2+xy^{-2}f_0^3 \tag{11}$$

Where $f_L=xy^2$

- Theorem VI.A.1

The enumerating function $f=f_0(x,y)$ satisfies the following equation $xf^3+3x^2y^2f^2+(3x^3y^4-y^2)f+x^4y^6=0$ (12)

Proof: According to LemmaA.1, A.2, A.3, and A.4, from $f_I=x^4y^4$ and (9), (10), (11), we have

$$f=x^4y^4+3x^2y^2f+2x^2f^2+(x^2f^2+xy^{-2}f^3)$$

(12) holds by grouping the terms. It is seen that f_0 is a solution of the equation when f_0 is in form as a power series of x,y .

- Theorem VI.A.2

The enumerating function $g=g_d(x,y)$ satisfies the following equation $xg^3+2x^2y^2g^2+(3x^3y^4-y^2)g+x^4y^6=0$ (13)

Proof: Similarly to the proof of Theorem A.1, we evaluate $g_d(x,y)$ excluding subcase1 in case 4.

Although from (12), (13), it is allowed to find $f_0(x,y)$ and $g_d(x,y)$ directly by employing Lagrangian inversion, because of the equations are triple, the result are rather complicated for usage. In order to find the enumerating explicit expressions, we have to do some transformation and find their parametric expressions firstly.

B. Parametric expressions and the determination of $g_d(x,y)$:

For the power series $g_d(x,y)$, we use the notations $\partial_{(x,y)}^{(i,j)}g_d(x,y)$ as the coefficients of x^iy^j in $g_d(x,y)$ .By dividing xf^3 the two sides of (5) and substituting $t=y^2$, we rewrite (13) after simplification as

$$\left(\frac{xt}{g}+1\right)^3=\left(\frac{xt}{g}\right)^2\bullet\frac{1}{x^3t}+\frac{xt}{g} \tag{14}$$

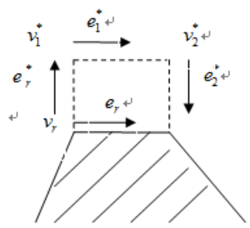

Fig. 4 Q_{II}

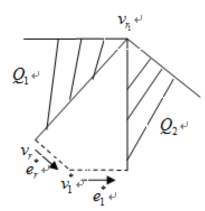

Fig.5 Q_{III}

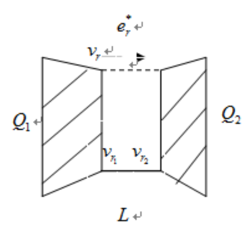

Fig.6 Q_{IV1}

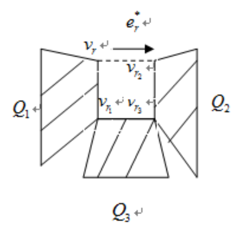

Fig.7 Q_{IV2}

Now we introduce two parameters θ and ξ . Let $\frac{xt}{f}=\frac{\xi+1}{\theta}$, then it is easy to very that the parametric expression of (14) is

$$\begin{cases} \frac{f}{xt}=\frac{\theta}{\xi+1},\ \xi=\frac{x^2}{1-\theta} \\ \theta=\frac{x^3t[(\xi+1+\theta)^3-\theta^2(\xi+1)]}{(\xi+1)^2} \end{cases} \tag{15}$$

- Theorem VI.B.1

The enumerating function $g=g_d(x,y)$ determined by equation (13) has the following expression as follows:

$$g_d(x,y)=x^4y^4+\sum_{m\ge 2,n\ge 0}\frac{(m-2)!}{(n+1)!n!i!j!k!(m-i-j-2)!}\times[2A_{i,j}B_{i,j}-A_{i+1,j-1}B_{i-1,j+1}]x^{3m+2n}y^{2m}$$

Where $A_{i,j}=\frac{(2m-2i-j-3)!}{(2m-k-2i-j-3)!}$; $B_{i,j}=\frac{(m+n-3i-j-k-4)!}{(m-n-3i-j-k-5)!}$

And $0\le 2i+j+k\le 2m-3$; $0\le 3i+j+k\le m-n-5$

Proof: By (15), we have the enumerating factor of $g=g_d(x,y)$, i.e., the matrix shown in [3], Theorem 1.5.2 as

$$\Delta_{(\theta,\xi)}=\begin{vmatrix} * & \frac{\xi[2\theta^2(1+\xi+\theta)-(1+\xi)^3]}{(1+\xi)[(1+\xi+\theta)^3-\theta^2(1+\xi)]} \\ \frac{\theta}{\theta-1} & 0 \end{vmatrix}$$
$$=\frac{\theta\xi[2\theta^2(1+\xi+\theta)-(1+\xi)^3]}{(1-\theta)(1+\xi)[(1+\xi+\theta)^3-\theta^2(1+\xi)]}$$

By applying Lagrangian inversion with two parameters, we have $\partial^{(m,n)}_{(x^3t,x^2)}(x^{-1}t^{-1}g)=\partial^{(m,n)}_{(\theta,\xi)}[(x^{-1}t^{-1}g)\phi_1^m\phi_2^n\Delta_{(\theta,\xi)}]$

Where $\phi_1=\frac{[(1+\xi+\theta)^3-\theta^2(1+\xi)]}{(1+\xi)^2}$ and $\phi_2=\frac{1}{1-\theta}$. Thus

$$\partial^{(m,n)}_{(x^3t,x^2)}(x^{-1}t^{-1}g)=\partial^{(m-2,n-1)}_{(\theta,\xi)}[(1+\xi+\theta)^3-\theta^2(1+\xi)]^{m-1}(1+\xi)^{-2m-2}$$
$$\times(1-\theta)^{-n-1}[2\theta^2((1+\xi+\theta)-(1+\xi)^3]$$

Finally by substituting m,n for respective $m+1,n-1$, we can derive the enumerating expressions of $g=g_d(x,y)$ described in Theorem VI.B.1.

For example, there are 63 rooted near quadrangulation maps on the disc with the size 11 and the valency of root-face is 6; there are 90 rooted near quadrangulation maps on the disc with the size 12 and the valency of root-face is 8.

C. Parametric expressions and explicit formulae for $f_0(x,y)$

By taking the same method of solving equation (13), (12) is in fact the form as

$$\left(\frac{xt}{f}\right)^2\frac{1}{x^3t}=\left(\frac{xt}{f}+1\right)^3 \tag{16}$$

Now, we introduce one parameter θ and let $\frac{xt}{f}=\frac{1-\theta}{\theta}$, then from (16), $f=f_0(x,y)$ has the parametric expressions as

$$\frac{f}{xt}=\frac{\theta}{1-\theta};\qquad \theta=\frac{x^3t}{(1-\theta)^2}$$

- Theorem VI. C.1

The enumerating function $f=f_0(x,y)$ determined by equation (12) has the following explicit expression:

$$f_0(x,y)=\sum_{n\ge 1}\frac{(3n)!}{n!(2n+1)!}x^{3n+1}y^{2n+2} \tag{17}$$

Proof: Here, based on the parametric expressions as described above, we enumerate the explicit expression of $f=f_0(x,y)$ by employing Lagrangian inversion with θ.

For example, there are 12 nonseparable rooted outerplanar quadrangulation maps with the size 10 and the valency of root-face is 8(See Fig. 8).

- Corollary VI. C.1

The number of nonseparable rooted outerplanar quadrangulation with the root-face of valency $2n$ and $3n-2$ edges (or $n-1$ internal faces, $n-2$ internal edges) for $n\ge 2$ is

$$C_n=\frac{(3n-3)!}{(n-1)!(2n-1)!}$$

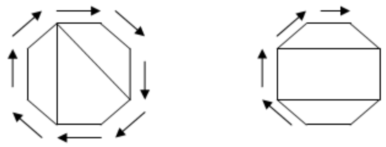

Fig. 8 Combinatorially distinct nonseparable rooted outerplanar quadrangulation

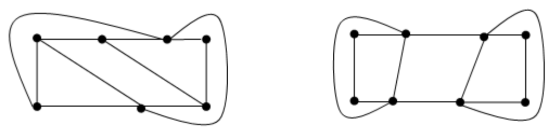

Fig. 9 Combinatorially distinct Hamiltonian planar quadrangulations

Proof: Based on (17) and Euler formula, a nonseparable rooted outerplanar quadrangulation of size $3n-2$ and order $2n$ has n faces. Taking into account of outerplanarity, the internal edges is 1 less than the internal faces, the corollary is done.

According to [6], the dual map of Q_0 are nonseparable rooted nearly 4-regular planar maps, the number of these maps with the root-vertex of valency $2m$ and l non-root faces (or non-root vertices) is $\sum_{k=l+1/2}^{l-m+2}2^{l-m-k+1}A_k(m,l)B_k(m,l)$, where

$$\begin{cases} A_k(m,l)=\frac{(k-m)(3m-3)!(2l-m-1)!}{(2m)!(m-1)!(l-m-k+2)!(2k-l)!(2l-k)!} \\ B_k(m,l)=4(2k-l)[m(k-m+1)-3j(2m-1)]+3j(j-1)(3m-2) \end{cases}$$

in which $j=l-m-k+2$. This result for nonseparable outerplanar map, we take $l=2n-1,m=n$, this is also Corollary VI. C.1.

Remark. Noticing that, for any nonseparable outerplanar map, there exist quadrangulations of the internal faces if and only if the valency of the root-face is even. Furthermore, any nonseparable outerplanar quadrangulation map is simple and bipartite.

VII. Rooted Hamiltonian planar quadrangulations

In this section, we find the relation between rooted nonseparable outerplanar quadrangulations and rooted Hamiltonian planar quadrangulations from the results obtained above.

Here are two nonseparable outerplanar quadrangulations with the same valency of root-face $2n$, so there are $n-1$ internal faces and $n-2$ internal edges for each one, but the partitions of vertices maybe different. After this, we make one boundary of the root-face adhere to the other, and then we gained a Hamiltonian planar quadrangulations which has $2n$ vertices, $2(n-1)$ faces and $2(n-2)$ edges not on the Hamilton circuit. Fig. 9 shows 2 combinatorial distinct Hamiltonian planar quadrangulations (with order 8).

VIII. Rooted (4, 3)-regular Halin map

In this section, we provide a one to one correspondence between rooted nonseparable outerplanar quadrangulations and rooted (4,3)-regular Halin map, and then find the number of rooted (4,3)-regular Halin map with the valency of the root-face and the edge number from the result obtained above.

Now we introduce a bijection, $\varphi: Q_0 \to H_{4,3}$, the set of rooted (4,3)-regular Halin map. In fact, for any $M\in Q_0$,we can obtain exactly $H\in H_{4,3}$ by following procedure:

Step1.Take an inner-point of each faces except the root-face of M and join the inner-points if the two corresponding faces have a common edge;

Step2.Then take the midpoint of each edge which is on the root-face and join a line between these midpoints and the inner-point of one face if the edges of the root-face are incident to this face;

Step3. Finally, add new edges to the midpoint of each root-face edge to form a cycle and remove the primary edges of this nonseparable outerplanar quadrangulations.

- Lemma VIII.1For $M\in Q_0$, $\varphi(M)=H$ and $H\in H_{4,3}$
- Lemma VIII.2 Let Q_n and H_m be the sets of all the rooted nonseparable outerplanar quadrangulations with n edges and the sets of all the rooted (4,3)-regular Halin map with m edges respectively, φ_{Q_n} be the restricted part of φ on Q_n , then φ_{Q_n} is a bijection between Q_n and H_m .
- Theorem VIII.1The number of rooted (4,3)-regular Halin map with the valency of root-face $2n$ and the edge number $5n-2$ is C_n , for $n\geq 2$.

Proof: From above mentioned steps, Lemma IV.1and IV.2, $H_{5n-2}\in H_{4,3}$ is formed by $M_{3n-2}\in Q_0$, for the theorem is proved.

Remark. For all vertices of rooted (4,3)-regular Halin map is 4 except the vertices on the root-face, the root-face valency is just the number of 3-valency root-face vertices, so if we add a multi-edge to each edge of root-face, the resultant map is 4-regular and is also nonseparable.

IX. Conclusions

In this paper, we provide functional equations satisfied by the enumerating functions for rooted planar near-quadrangulations with the size, the valency of the root-face and the number of non-rooted vertices and rooted nonseparable outerplanar quadrangulations dependent on the size and the valency of the root-face respectively. For the equations are quadruple or cubic, the results are rather complicated for usage. In order to find the enumerating explicit expressions, we first do some transformations and then find their parametric expressions. Finally, explicit formulae for such these types of quadrangulations with above parameters are derived by employing Lagrangian inversion based on the enunfunctions. Furthermore, we present a summation-free formula for rooted nonseparable outerplanar quadrangulations. As consequences, the numbers of rooted planar trees and outerplanar quadrangulations, rooted Hamiltonian planar quadrangulations with even order and rooted (4,3)-regular Halin map are extracted more directly and more simply.

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